Conservative Reduced Order Modeling of the Plasma Kinetic Equations

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Abstract

- Recent interest in spectral methods for fusion, space, and astrophysical plasma simulations ⇒ **noiseless (gyro-)kinetic simulations.**
- We propose a data-driven projection-based reduced-order model (ROM) to reduce the computational cost of the spectral plasma solver (SPS) [1] of the Vlasov-Poisson equations, describing the equations of motion of collisionless electrostatic plasma.
- **Contribution:** ROM for the kinetic effects (higher order moments) in SPS while keeping the macroscopic equations intact
  ⇒ **preservation of fluid-kinetic property, conserves mass, momentum, and energy, and efficiently handles convolutions.**

Vlasov-Poisson equations

We consider the Vlasov-Poisson collisionless equations

$$\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x} + \frac{q^s}{m^s}E(x,t)\frac{\partial}{\partial v}\right) f^s(x,v,t) = 0 \qquad \& \qquad \frac{\partial E(x,t)}{\partial x} = \sum_s q^s \int f^s(x,v,t)\mathrm{d}v.$$

The conserved quantities in a periodic spatial domain are

$$\mathcal{N}^s(t) = \int\int f^s(x,v,t)\mathrm{d}v\mathrm{d}x \qquad \text{(mass)}$$

$$\mathcal{P}(t) = \sum_s m^s \int\int v f^s(x,v,t)\mathrm{d}v\mathrm{d}x \qquad \text{(momentum)}$$

$$\mathcal{E}(t) = \frac{1}{2}\int E(x,t)^2\mathrm{d}x + \sum_s \frac{m^s}{2}\int\int v^2 f^s(x,v,t)\mathrm{d}v\mathrm{d}x \qquad \text{(energy)}$$

and infinitely many more...

Spectral Plasma Solver [1]

Hermite-Fourier spectral expansion

$$f^s(x,v,t) \approx \sum_{n=0}^{N_v-1}\sum_{k=-N_x}^{N_x} \psi_n(\xi^s(v))\exp\left(\frac{2\pi i k}{\ell}\right)C_n^s(t) \qquad \& \qquad E(x,t) \approx \sum_{k=-N_x}^{N_x}\exp\left(\frac{2\pi i k}{\ell}\right)E_k(t)$$

- **Velocity** coordinate is projected onto the asymmetrically weighted **Hermite basis** ⇒ fast convergence for near-Maxwellian plasma
  $$\psi_n(\xi^s) = (\pi 2^n n!)^{-\frac{1}{2}}\mathcal{H}_n(\xi^s)\exp\left(-(\xi^s)^2\right) \qquad \text{and} \qquad \xi^s(v) = (v-u^s)/\alpha^s.$$
- **Spatial** coordinate is projected onto the **Fourier basis** ⇒ spatial waves in periodic domain.
- Asymmetric Hermite approach possesses a simple correspondence between spectral coefficients and fluid moments similar to Grad 1949 [2] ⇒ **fluid-kinetic** property.
- **Linear and quadratic invariants** of the system can be conserved at the fully discrete level using an implicit Gauss-Legendre temporal integrator.
- We use closure by truncation $C^s_{N_v}(t) = 0$ along with a (hyper) Lenard-Bernstein collisional operator [3] to handle filamentation and numerical recurrence effects.

Fluid-kinetic proper orthogonal decomposition

The semi-discrete SPS Vlasov-Poisson equations in vector form can be written as

$$\frac{\mathrm{d}\mathbf{C}_F^s}{\mathrm{d}t} = \mathbf{A}_F^s\mathbf{C}_F^s(t) + \mathbf{B}_F^s\boldsymbol{\mathcal{N}}\left(\mathbf{C}_F^s(t),\mathbf{E}(t)\right) + \mathbf{G}_F^s\mathbf{C}_K^s(t), \tag{1}$$

$$\frac{\mathrm{d}\mathbf{C}_K^s}{\mathrm{d}t} = \mathbf{A}_K^s\mathbf{C}_K^s(t) + \mathbf{B}_K^s\boldsymbol{\mathcal{N}}\left(\mathbf{C}_K^s(t),\mathbf{E}(t)\right) + \mathbf{G}_K^s\mathbf{C}_F^s(t) + \mathbf{J}_K^s\left[\mathbf{E}(t) * \boldsymbol{\Theta}_F\mathbf{C}_F^s(t)\right], \tag{2}$$

$$\mathbf{E}(t) = \mathbf{D}\sum_s q^s\alpha^s\mathbf{C}_F^s(t), \tag{3}$$

where $\mathbf{C}_F^s(t)$ and $\mathbf{C}_K^s(t)$ are the **fluid and kinetic state vectors** of species s and $\boldsymbol{\mathcal{N}}$ is a convolution operator between the electric field and Hermite moments.

We assume there is a low-dimensional representation of the kinetic state vector

$$\mathbf{C}_K^s(t) \approx \mathbf{U}_r^s\mathbf{C}_{K,r}^s(t) \qquad \text{s.t.} \qquad (\mathbf{U}_r^s)^*\mathbf{U}_r^s = \mathbf{I}_{N_r}, \qquad \mathbf{U}_r^s \in \mathbb{C}^{N_K\times N_r}, \qquad N_r \ll N_K,$$

where $\mathbf{U}_r^s$ is the POD basis. Galerkin projection applied to Eq. (2) yields the ROM kinetic equation

$$\frac{\mathrm{d}\mathbf{C}_{K,r}^s}{\mathrm{d}t} = \mathbf{A}_{K,r}^s\mathbf{C}_{K,r}^s(t) + \mathbf{B}_{K,r}^s\left[\mathbf{C}_{K,r}^s(t)\otimes\mathbf{E}(t)\right] + \mathbf{G}_{K,r}^s\mathbf{C}_F^s(t) + \mathbf{J}_{K,r}^s\left[\mathbf{E}(t) * \boldsymbol{\Theta}_F\mathbf{C}_F^s(t)\right]. \tag{4}$$

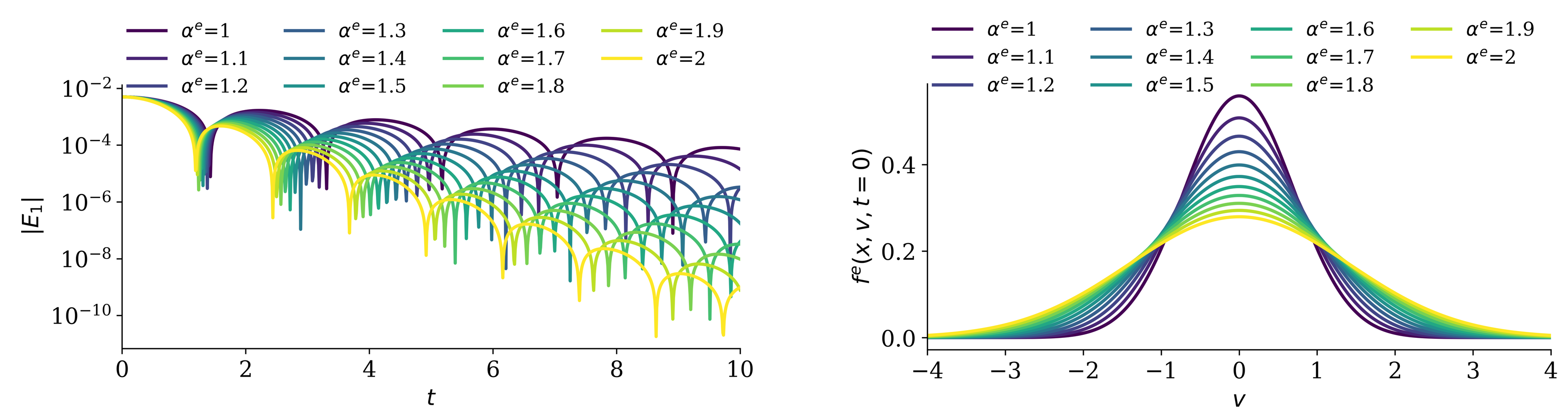
Arithmetic complexity: Cooley-Tukey FFT vs. Kronecker product

FOM operation	algebraic term	arithmetic complexity
linear advection	$\mathbf{A}_K^s\mathbf{C}_K^s(t)$	$\mathcal{O}(N_K)$
nonlinear acceleration	$\mathbf{B}_K^s\boldsymbol{\mathcal{N}}\left(\mathbf{E}(t),\mathbf{C}_K^s(t)\right)$	$\mathcal{O}(N_K\log_2(N_{x,t}))$
linear coupling to kinetic moments	$\mathbf{G}_K^s\mathbf{C}_K^s(t)$	$\mathcal{O}(N_{x,t})$
nonlinear coupling to kinetic moments	$\mathbf{J}_K^s[\mathbf{E}(t) * \boldsymbol{\Theta}_F^\top\mathbf{C}_F^s(t)]$	$\mathcal{O}(N_{x,t}\log_2(N_{x,t}))$
total evaluation	Eq. (2)	$\mathcal{O}(N_K\log_2(N_{x,t}))$

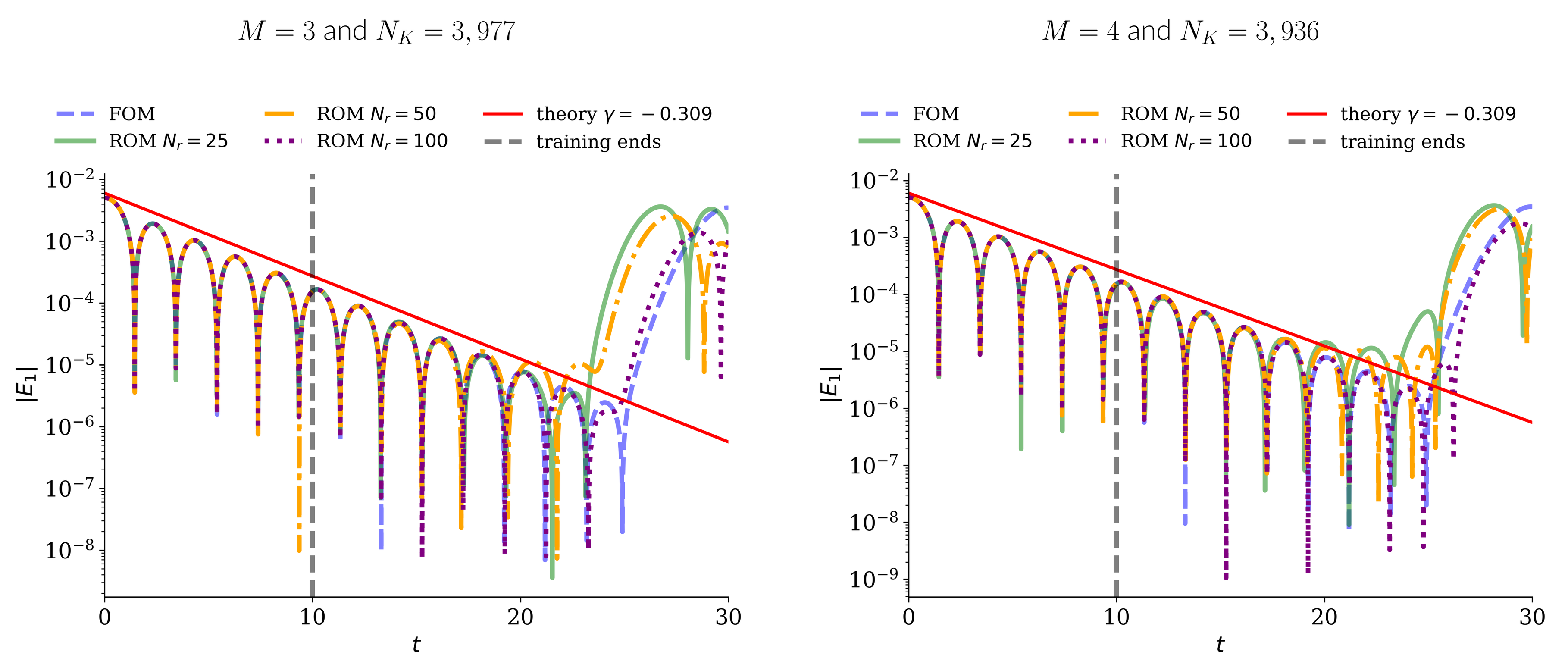
ROM operation	algebraic term	arithmetic complexity
linear advection	$\mathbf{A}_{K,r}^s\mathbf{C}_{K,r}^s(t)$	$\mathcal{O}(N_r^2)$
nonlinear acceleration	$\mathbf{B}_{K,r}^s\left[\mathbf{C}_{K,r}^s(t)\otimes\mathbf{E}(t)\right]$	$\mathcal{O}(N_r^2 N_{x,t})$
linear coupling to fluid moments	$\mathbf{G}_{K,r}^s\mathbf{C}_F^s(t)$	$\mathcal{O}(N_{x,t}N_r)$
nonlinear coupling to fluid moments	$\mathbf{J}_{K,r}^s\left[\mathbf{E}(t) * \boldsymbol{\Theta}_F^\top\mathbf{C}_F^s(t)\right]$	$\mathcal{O}(N_{x,t}\log_2(N_{x,t}) + N_r N_{x,t})$
total evaluation	Eq. (4)	$\mathcal{O}(N_r^2 N_{x,t})$

Test 1: weak Landau damping

Training parametric simulations $N_v = 100$, $N_x = 20$, $\Delta t = 0.01$ with 11 samples of α^e

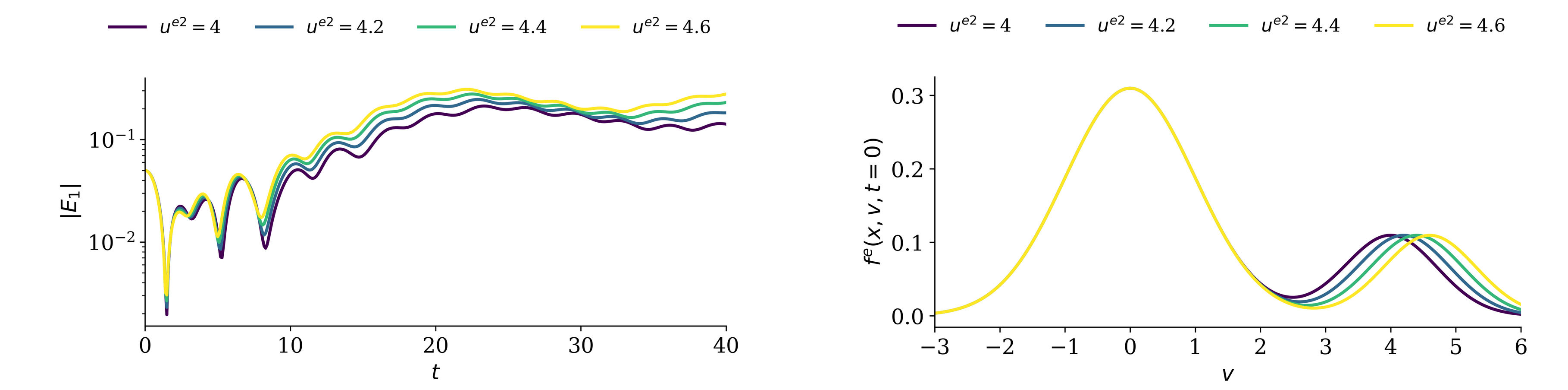


Extrapolation: ROM prediction for thermal velocity $\alpha^e = 0.9$

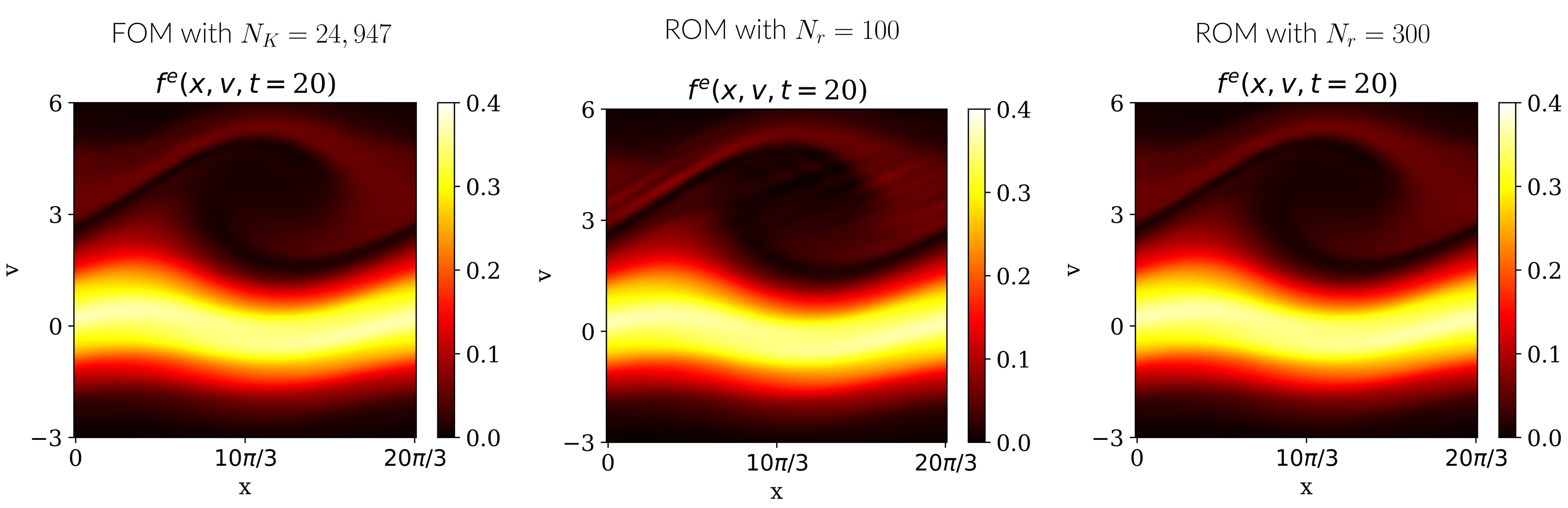


Test 2: bump-on-tail instability

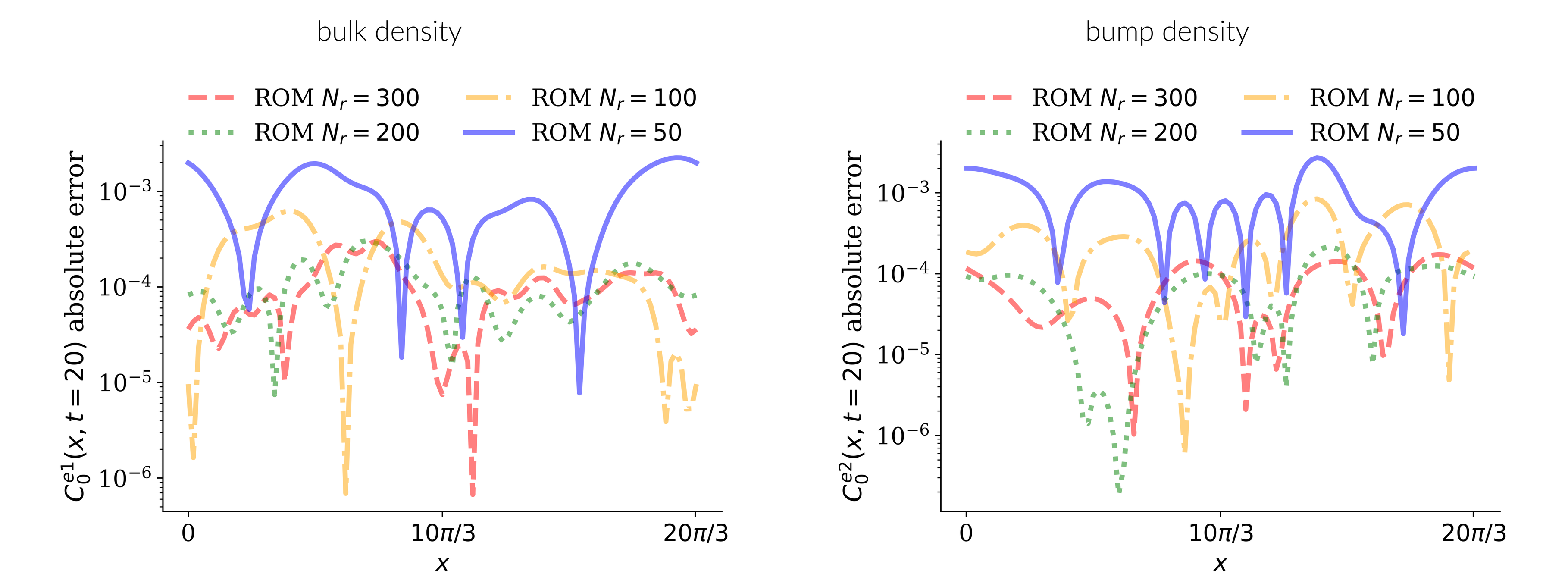
Training parametric simulations $N_v = 250$, $N_x = 50$, $\Delta t = 0.01$ with 4 samples of u^{e2}



Interpolation: ROM prediction for bump bulk speed $u^{e2} = 4.5$



How are the macroscopic quantities affected?



Conclusions

Benchmark cases verify the properties of the proposed fluid-kinetic ROM approach

- ✓ Mass/momentum/energy conservation to machine precision
- ✓ Efficiently handling convolutions in reduced dimensions via Kronecker products
- × Accurate predictions in nonlinear regimes (e.g. bump-on-tail $t > 20$) needs further investigation

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