Anti-symmetric spectral moments of the Vlasov-Poisson equations

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Abstract

- Recent interest in spectral methods for fusion and astrophysical plasma simulations $\Rightarrow$ noiseless (gyro-)kinetic simulations.
- We propose a symmetrically weighted Hermite spectral (in velocity) and central finite difference (in space) discretization that preserves the anti-symmetric structure of the advection operator in the 1D1V Vlasov equation $\Rightarrow$ unconditionally stable numerical method.
- We apply such discretization to two formulations: the canonical Vlasov-Poisson equations and their continuously transformed square-root representation $\Rightarrow$ square-root preserves the positivity of the particle distribution function.

Vlasov-Poisson equations

As a starting point, we consider the Vlasov-Poisson collisionless equations

$$\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x} + \frac{q^s}{m^s}E(x,t)\frac{\partial}{\partial v}\right) f^s(x,v,t) = 0 \qquad \text{and} \qquad \frac{\partial E(x,t)}{\partial x} = \sum_s q^s \int f^s(x,v,t)\mathrm{d}v.$$

The conserved quantities in a periodic spatial domain are

$$\mathcal{N}^s(t) = \int\int f^s(x,v,t)\mathrm{d}v\mathrm{d}x \qquad \text{(mass)}$$

$$\mathcal{P}(t) = \sum_s m^s \int\int v f^s(x,v,t)\mathrm{d}v\mathrm{d}x \qquad \text{(momentum)}$$

$$\mathcal{E}(t) = \frac{1}{2}\int E(x,t)^2\mathrm{d}x + \sum_s \frac{m^s}{2}\int\int v^2 f^s(x,v,t)\mathrm{d}v\mathrm{d}x \qquad \text{(energy)}$$

$$\mathcal{L}^p(t) = \int\int (f^s(x,v,t))^p\mathrm{d}v\mathrm{d}x \qquad (L^p \text{ norm})$$

and more...

Anti-symmetric Hermite spectral moments [1]

Expansion of f^s

$$f^s(x,v,t) \approx \sum_{n=0}^{N_v-1} C_n^s(x,t)\psi_n(\xi^s),$$

Expansion of $\sqrt{f^s}$

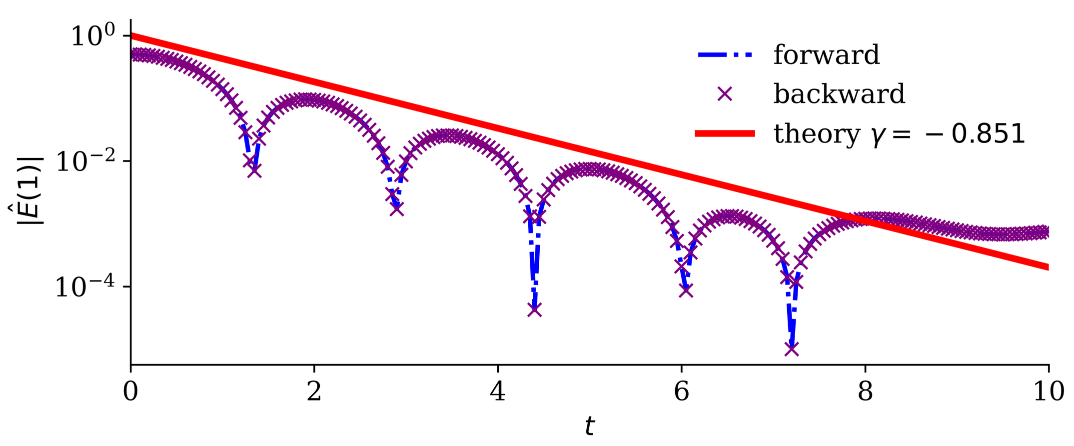
$$\sqrt{f^s(x,v,t)} \approx \sum_{n=0}^{N_v-1} C_n^s(x,t)\psi_n(\xi^s).$$

- The velocity coordinate is projected onto the symmetrically weighted Hermite basis
$$\psi_n(\xi^s) = (\sqrt{\pi}2^n n!)^{-\frac{1}{2}}\mathcal{H}_n(\xi^s)\exp\left(-(\xi^s)^2/2\right) \qquad \text{and} \qquad \xi^s(v) = (v-u^s)/\alpha^s.$$
- The spatial coordinate is discretized via central finite differencing $\Rightarrow$ anti-symmetric derivative operator $\mathbf{D}_x = -\mathbf{D}_x^\top$.
- Both the f^s and $\sqrt{f^s}$ formulations result in the same discretized system with different conservation properties.
- The anti-symmetric approach does not possess a simple correspondence between spectral coefficients and fluid moments as Grad 1949 [3] or the asymmetrically weighted Hermite spectral approach [4].
- Linear and quadratic invariants of the system can be conserved at the fully discrete level using an implicit Gauss-Legendre temporal integrator.
- Closure by truncation $C_{N_v}^s(x,t) = 0$ is the most appropriate conservative closure for the f^s formulation [2].

Why is it important to preserve anti-symmetry?

- An anti-symmetric operator $A = -A^\top$ conserves square norms since
$$\phi^\top A\phi = -\phi^\top A\phi = 0$$
- Vlasov advection can be expressed in anti-symmetric form
$$\frac{\partial}{\partial t}f^s(x,v,t) = A^s(x,v,t)f^s(x,v,t) \quad \Rightarrow \quad A^s(x,v,t) = -v\frac{\partial}{\partial x} - \frac{q^s}{m^s}E(x,t)\frac{\partial}{\partial v}$$
- There is an equivalent advection operator in discrete form
$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\Psi}^s(t) = \mathbf{A}^s(t)\boldsymbol{\Psi}(t) \quad \Rightarrow \quad \mathbf{A}^s(t) = -\mathbf{V}\otimes\mathbf{D}_x - \frac{q^s}{m^s}\mathbf{E}(t)\otimes\mathbf{D}_v$$

An anti-symmetric structure-preserving discretization is unconditionally stable and renders explicit temporal integrators approximately time-reversible [1].

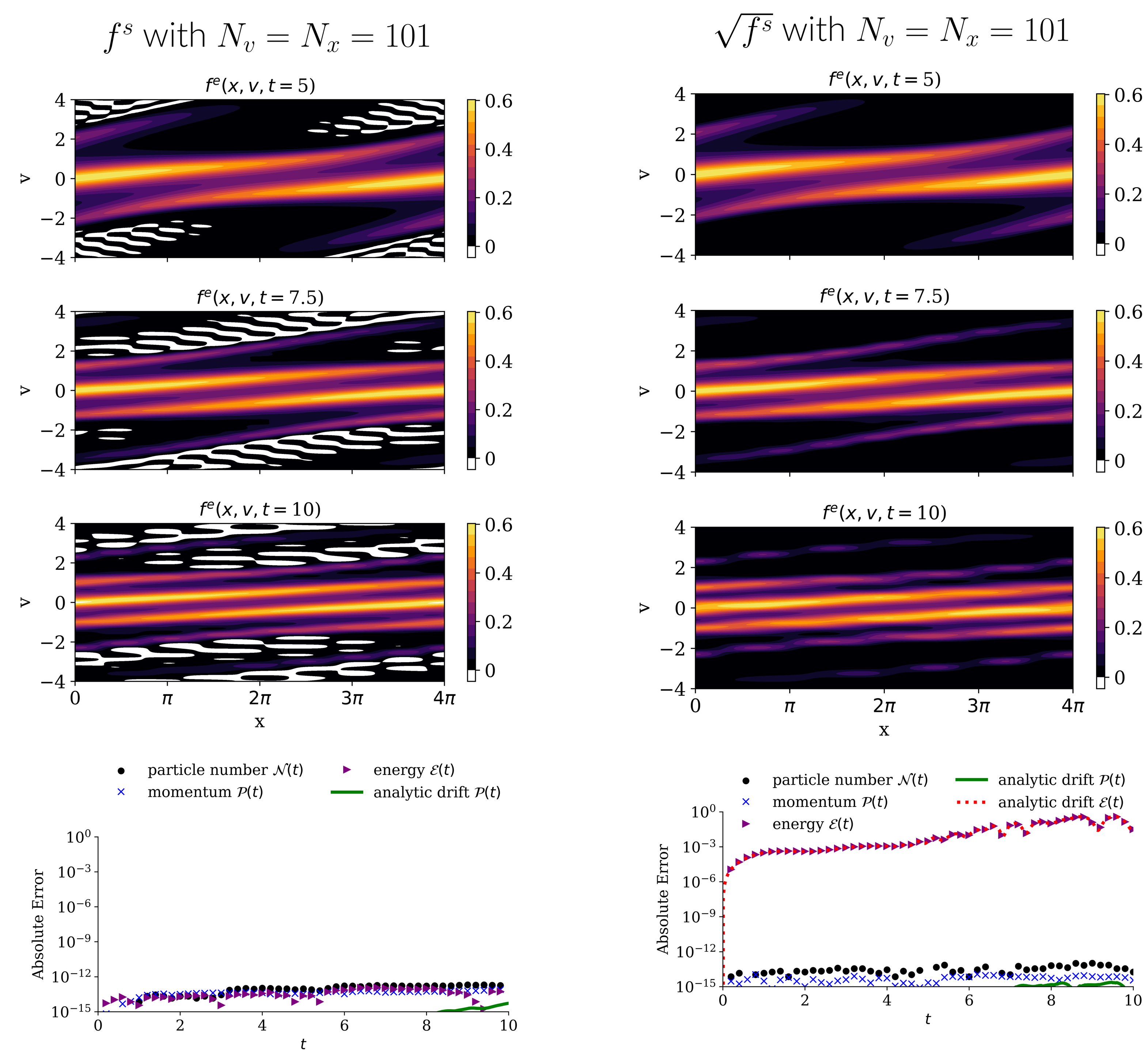


Linear Landau damping with RK3

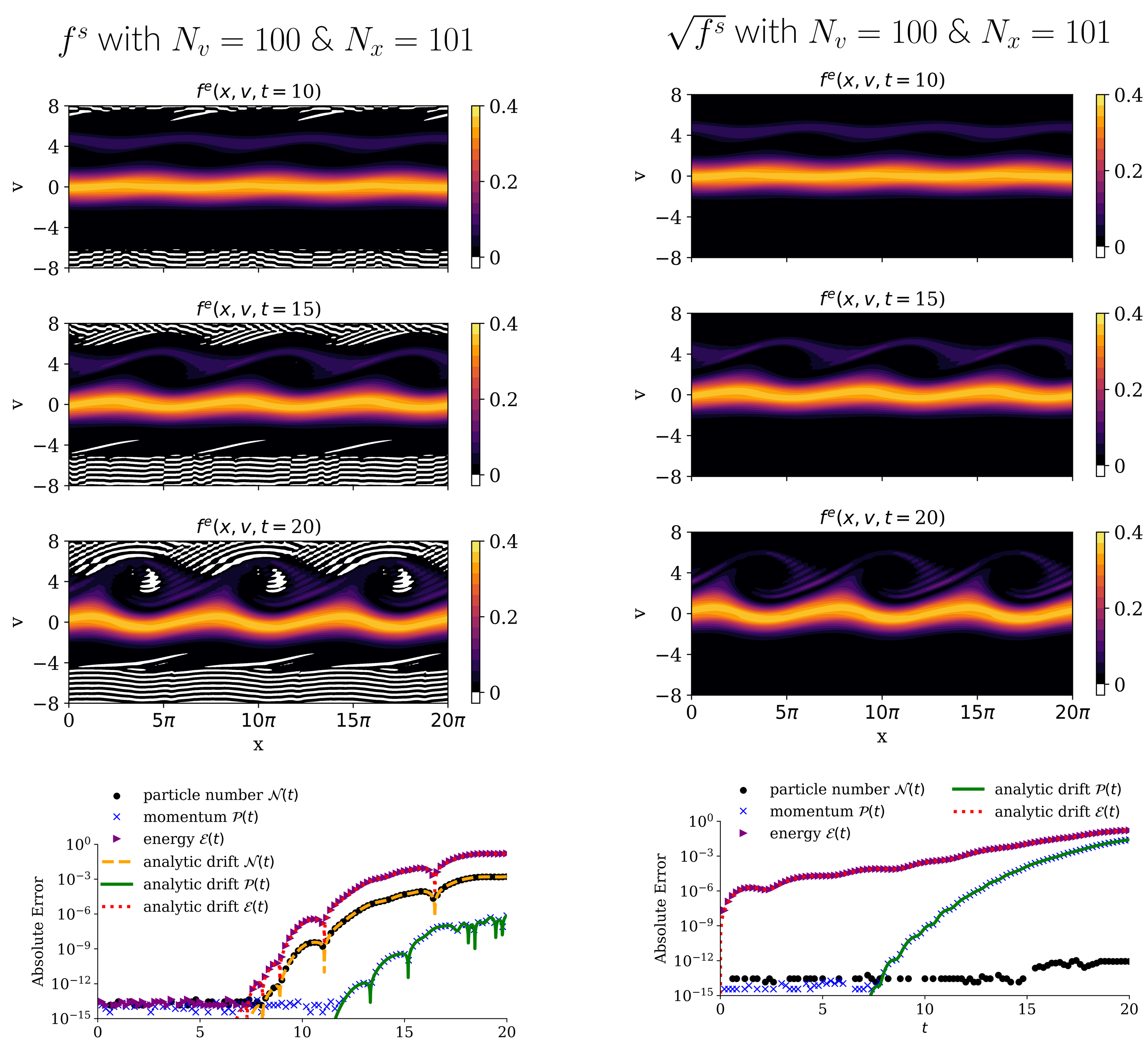
Numerical properties of f^s vs. $\sqrt{f^s}$

	f^s	$\sqrt{f^s}$
conservation of mass	if N_v is odd	✓
conservation of momentum	if N_v is even & $u^s = 0, \forall s$	✗
conservation of energy	if N_v is odd & $u^s = 0, \forall s$	✗
positivity preserving	✗	✓
unconditionally stable	✓	✓

Nonlinear Landau damping



Bump-on-tail instability



Conclusions

- Benchmark cases verify properties of the anti-symmetric approach
  - ✓ $(\sqrt{f^s})^2$ and $(f^s)^2$ are quadratic invariant of the discrete system $\Rightarrow$ unconditionally stable numerical scheme
  - ✓ Mass/momentum/energy conservation to machine precision depending on N_v and u^s for f^s formulation
  - ✓ Mass conservation to machine precision for $\sqrt{f^s}$ formulation
  - × Manageable momentum and energy drifts
- Future work avenues
  - (1) Develop techniques to mitigate filamentation while maintaining conservation
  - (2) Adaptivity in time and space of the Hermite parameters u^s and α^s

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